Announcements

1) Proof that

$\det(AB) = \det(A)\det(B)$

is on 11/14 notes

Theorem: (det properties, cont.)

4) $\det(A^t) = \det(A)$

5) If B is obtained from A by adding a multiple of a row of A to another row, then $\det(B) = \det(A)$

6) $\boxed{\det(A) \neq 0 \text{ if and only if } A \text{ is invertible}}$

7) If B is obtained from A by multiplying a row of A by $\alpha \in \mathbb{C}$,

$$\det(B) = \alpha \det(A)$$

8) If B is obtained from A by interchanging two rows of A,

$$\det(B) = -\det(A)$$

proof: 6) assuming 4) and 5)

$\Leftarrow$ Assume A is invertible.

By property 1) from last class,

$$\boxed{\begin{aligned} 1 = \det(I_n) &= \det(A \cdot A^{-1}) \\ &= \det(A) \cdot \det(A^{-1}) \end{aligned}}$$

by property 3). Hence,

$\det(A) \neq 0$.

$\Rightarrow$ Use the contrapositive and assume A is not invertible. What we want to show is

$$\boxed{\det(A) = 0}$$

If A is not invertible, then $\ker(A) \neq \{0_{\mathbb{R}^n}\}$.

Take $h \in \ker(A)$, $h \neq 0_{\mathbb{R}^n}$.

Write $h = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{bmatrix}$, $A = (a_{i,j})_{i,j=1}^{n}$

Then

$$A\left(\begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Unraveling the action of A on h,

$\forall \quad i, 1 \leq i \leq n,$

$$\sum_{j=1}^{n} a_{i,j} h_j = 0. \text{ Then}$$

$$h_1 \begin{bmatrix} a_{1,1} \\ a_{2,1} \\ \vdots \\ a_{n,1} \end{bmatrix} + h_2 \begin{bmatrix} a_{1,2} \\ a_{2,2} \\ \vdots \\ a_{n,2} \end{bmatrix} + \cdots + h_n \begin{bmatrix} a_{1,n} \\ a_{2,n} \\ \vdots \\ a_{n,n} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Since $h \neq 0$, $\exists$ j, $1 \leq j \leq n$,

with $h_j \neq 0$. Then

$$h_j \begin{bmatrix} a_{1,j} \\ a_{2,j} \\ \vdots \\ a_{n,j} \end{bmatrix} = -\sum_{\substack{k=1 \\ k \neq j}}^{n} h_k \begin{bmatrix} a_{k,1} \\ a_{k,2} \\ \vdots \\ a_{k,n} \end{bmatrix}$$

Dividing by h_j, we get

$$\boxed{\begin{bmatrix} a_{1,j} \\ a_{2,j} \\ \vdots \\ a_{n,j} \end{bmatrix} = \sum_{\substack{k=1 \\ k \neq j}}^{n} \frac{h_k}{h_j} \begin{bmatrix} a_{1,k} \\ a_{2,k} \\ \vdots \\ a_{n,k} \end{bmatrix}}$$

This says the j^{th} column of A is a linear combination of the other columns of A.

This says the j^{th} row of A^t is a linear combination of the other rows of A^t. Using 5) repeatedly, by adding $\left(\frac{h_k}{h_j}\right.$ times the k^{th} row of $\left.A^t\right)$, we obtain a matrix B with

$$\det(B) = \det(A^t)$$

$$= \det(A) \text{ by } 4).$$

But if $B = (b_{i,k})_{i,k=1}^{n}$,

the j^{th} row of B has

all zero entries. Then

$$\det(B) = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \prod_{i=1}^{n} b_{i,\sigma(i)}$$

$$= \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \underbrace{b_{j,\sigma(j)}}_{\substack{=0 \\ \forall \sigma \in S_n}} \prod_{\substack{i=1 \\ i \neq j}}^{n} b_{i,\sigma(i)}$$

$$= 0.$$

Since $0 = \det(B) = \det(A)$,

we are done.

5) Let $\alpha \in \mathbb{C}$.

Let $1 \leq s, t \leq n$, $s \neq t$.

Let $A = (a_{i,j})_{i,j=1}^{n}$ and

let B be the matrix

with $\boxed{b_{i,j} = a_{i,j}}$ $\forall\, i \neq t$

and $\boxed{b_{t,j} = a_{t,j} + \alpha a_{s,j}}$

Then

$$\det(B)$$

$$= \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \prod_{i=1}^{n} b_{i,\sigma(i)}$$

$$= \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \prod_{\substack{i=1 \\ i \neq t}}^{n} a_{i,\sigma(i)} \cdot \left(\overbrace{a_{t,\sigma(t)} + \alpha a_{s,\sigma(t)}}^{b_{t,\sigma(t)}} \right)$$

$$= \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)} \quad \left(= \det(A) \right)$$

$$+ \sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\, \alpha \prod_{\substack{i=1 \\ i \neq s,t}}^{n} a_{i,\sigma(i)} \left(a_{s,\sigma(s)}\right)\left(a_{s,\sigma(t)}\right)$$

For each $\sigma \in S_n$,

if $\gamma = \sigma \circ (st)$,

then $\text{sign}(\gamma) = \text{sign}(\sigma) \cdot \text{sign}((st))$

$= -\text{sign}(\sigma)$.

Moreover,

$$\prod_{\substack{i=1 \\ i \neq s,t}}^{n} a_{i,\gamma(i)} (a_{s,\gamma(s)})(a_{s,\gamma(t)})$$

$$= \prod_{\substack{i=1 \\ i \neq s,t}}^{n} a_{i,\sigma(i)} (a_{s,\sigma(t)})(a_{s,\sigma(s)})$$

So this γ term cancels out the σ term in the sum. Since this pairing is 1-1, we get

$$\det(B) = \det(A).$$

4) Let $A = (a_{i,j})_{i,j=1}^{n}$.

Let $A^t = B = (b_{i,j})_{i,j=1}^{n}$.

Then

$$\det(B) = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \prod_{i=1}^{n} b_{i,\sigma(i)}$$

$$= \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \prod_{i=1}^{n} b_{\sigma^{-1}(\sigma(i)),\sigma(i)}$$

Now if we let $j = \sigma(i)$, then since σ is a bijection, j runs from 1 to n, and the sum becomes

$$\sum_{\sigma \in S_n} \text{sign}(\sigma) \prod_{j=1}^{n} b_{\sigma^{-1}(j),j}$$

$$= \sum_{\sigma \in S_n} \text{sign}(\sigma^{-1}) \prod_{j=1}^{n} b_{\sigma^{-1}(j),j}$$

(since $\text{sign}(\sigma) = \text{sign}(\sigma^{-1})$)

$$= \sum_{\sigma^{-1} \in S_n} \text{sign}(\sigma^{-1}) \prod_{j=1}^{n} b_{\sigma^{-1}(j),j}$$

Now with $\gamma = \sigma^{-1}$, the sum is

$$\sum_{\gamma \in S_n} \text{sign}(\gamma) \prod_{j=1}^{n} b_{\gamma(j),j}$$

$$= \sum_{\gamma \in S_n} \text{sign}(\gamma) \prod_{j=1}^{n} a_{j,\sigma(j)}$$

$$= \det(A)$$

7) and 8)

exercises